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Mapping Properties For Conic Regions Associated With Rabotnov Functions

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Abstract

In this paper, Our focus is to obtain the appropriate conditions for the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t}) = \mathbb{R}_{\alpha,\beta}(\mathrm{t}) * \mathcal{H}(\mathrm{t})$ that belongs to actual classes $UCV(\mathcal{K}, \tilde{\alpha})$, $S_p(\mathcal{K}, \tilde{\alpha})$ and $S^*{}_{ʒ}$

Keywords:

Analytic Function, Univalent Function, Starlike function, Convex Function, Uniformly Convex.

Introduction and Beginnings

Assume that $\hat{A}$ is the category of functions of type

$$\mathcal{H}(t) = t + \sum_{n=2}^{\infty} a_n t^n , \quad (1)$$

All functions in ʂ that are univalent in U fall into the class S, and $\mathcal{H}(t)$ is analytic in the open unit disc $\mathrm{U} = \{\mathrm{t} : |\mathrm{t}| < 1\}$.The classes of starlike and convex functions of order α are delimited by $S^*(\tilde{\alpha})$ and $\acute{\mathcal{C}}(\tilde{\alpha})$, respectively, and are defined as follows:

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$$S^*(\tilde{\alpha}) = \{ \mathcal{H}: \mathcal{H} \in \hat{A} \text{ and } \mathcal{R}\left(\frac{t\mathcal{H}'(t)}{\mathcal{H}(t)} \right) > \tilde{\alpha},\ t \in \mathrm{U}, \tilde{\alpha} \in [0,1]$$

and

$$\acute{\mathcal{C}}(\tilde{\alpha}) = \{ \mathcal{H}: \mathcal{H} \in \hat{A} \text{ and } \mathcal{R}\left(1 + \frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)} \right) > \tilde{\alpha},\ t \in \mathrm{U}, \tilde{\alpha} \in [0,1]$$

this is evident that

$$S^*(0) = S^* \text{ and } \acute{\mathcal{C}}(0) = \acute{\mathcal{C}}$$

These classes were firstly showed by, The Robertson in (1936), see [1, 2] for further details. Uniformly starlike functions (S_p) and uniformly convex functions (UCV) were first proposed by Goodman [3, 4] in 1991. If the mapping of $\mathcal{H}(\tau)$ is convex for any circular arc τ that is contained in the (OUD) open unit disc, it also has the center in it. then the function $\mathcal{H} \in \hat{A}$ is uniformly convex(UC).

In 1992 Minda and Ma [5] and in 1993 Ronning [6] showed that freely:

1.1(Uniformly Convex Function)

In the open unit disc, a function $\mathcal{H} \in \hat{A}$ is uniformly convex if and only if

$$\mathcal{R}\left(1 + \frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}\right) > \left|\frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}\right| . \qquad (2)$$

Alternatively, if $1 + \frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}$ lies in the parabolic area, In the open unit disc, we can state that a function $\mathcal{H} \in \hat{A}$ is uniformly convex (UC).

1.2(Ꝁ−Uniformly Convex Functions)

A function $\mathcal{H} \in \hat{A}$ is in S_p if

$$\mathcal{R}\left(\frac{t\mathcal{H}'(t)}{\mathcal{H}(t)} \right) > \left|\frac{t\mathcal{H}'(t)}{\mathcal{H}(t)} - 1\right| . \qquad (3)$$

The subcategories of Ꝁ −uniformly convex functions of order α and a new category related with the starlike functions are then shown. Bharati et al. defined these groups in (1997) [7] explained as follows:

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Definition 1.3 A function $\mathcal{H} \in \hat{A}$ is in UCV (Ҟ, α̃) if and only if

$$\mathcal{R}\left(1+\frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}\right) > Ҟ \left|\frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}\right| + \tilde{\alpha},\ \ t \in U, \qquad (4)$$

where $0 \leq Ҟ < \infty$ and $0 \leq \tilde{\alpha} < 1$.

The class $S_p(Ҟ, \tilde{\alpha})$ defined can be built by using the Alexander transform as follows:

Definition 1.4 A function $\mathcal{H} \in UCV\,(Ҟ, \tilde{\alpha})$ "if and only if" $t\mathcal{H}' \in S_p(Ҟ, \tilde{\alpha})$

In 1997 Ronning and Ponnusamy [8] were hosted the classes $C_£$ and $S^*_£$. These classes shown as follow:

Definition 1.5 A function $\mathcal{H} \in \hat{A}$ and

$$\left|\frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)}\right| < £,\ (t \in U, £ > 0), \qquad (5)$$

then $\mathcal{H} \in C_£$.

Definition 1.6 A function $\mathcal{H} \in \hat{A}$ and

(6)

$$\left|\frac{t\mathcal{H}'(t)}{\mathcal{H}(t)} - 1\right| < £,\ (t \in U, £ > 0),$$

then $\mathcal{H} \in S^*_£$.

In 2004 Swaminathan [9], was presented a class $Ғ^{\tau(ɲ)}_{ǿ}$. This class will perform a very significant role in our main outputs. The class $Ғ^{\tau(ɲ)}_{ǿ}$ is defined as:

Definition 1.7 A function $\mathcal{H} \in \hat{A}$ and fulfils

$$\left|\frac{(1-ǿ)\frac{\mathcal{H}(t)}{t} + ǿ\mathcal{H}'(t) - 1}{2\tau(1-ɲ) + (1-ǿ)\frac{\mathcal{H}(t)}{t} + ǿ\mathcal{H}'(t) - 1}\right| < 1, \qquad (7)$$

where $ɲ < 1$ and τ , $ǿ \in [0,1)$ belongs (∈) to the complex numbers except 0, then $\in Ғ^{\tau(ɲ)}_{ǿ}$.

Remark 1.8. If $\tau = iƀ \cos ƀ$, for $ƀ \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the class $Ғ^{\tau(ɲ)}_{ǿ}$ can also be defined as:

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$$F^{\tau(\acute{\eta})}_{\acute{\phi}} = \{ \mathcal{H} \in \hat{A} : \mathcal{R}\left\{e^{i\sigma}(1-\acute{\phi})\frac{\mathcal{H}(t)}{t} + \acute{\phi}\mathcal{H}'(t) - \acute{\eta}\right\} > 0 , \sigma \in \mathcal{R} . \quad (8)$$

Yu. N. Rabotnov in 1948 [10] presented a unique function used in viscoelasticity. Rabotnov's discoveries to solid mechanics included nonelastic stability, plasticity, shell theory, genetics, failure mechanics, creep theory, and composites. The Rabotnov function, also referred to as the Rabotnov fractional exponential function or simply the Rabotnov function, has the following definition.

$$\mathbb{R}_{\tilde{\alpha},\beta}(t) = t^{\tilde{\alpha}} \sum_{m=0}^{\infty} \frac{\beta^m \quad t^{m(1+\tilde{\alpha})}}{\Gamma((m+1)(1+\tilde{\alpha}))} \quad (9)$$

We discussed on some of the geometric features of the following normalized form of Rabotnov function.

$$\mathbb{R}_{\tilde{\alpha},\beta}(t) = t + \sum_{m=2}^{\infty} \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} \quad t^m . \quad (10)$$

1.9 Convolution

Applications for the Hadamard or Convolution product have multiple uses in the field of geometric function theory. This section provides a definition for the term convolution.

Definition 1.10 The Hadamard convolution of the functions of class Ȧ is defined by

$(\mathcal{H} * g)t = t + \sum_{m=1}^{\infty} a_m b_m t^m \quad (t \in U)$

here $\mathcal{H}(t)$ and $g(t)$ are convergent power series in the open unit disc.

Now, we discuss about the importance of convolution operator

$$\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) = \mathbb{R}_{\alpha,\beta}(t) * \mathcal{H}(t)$$

$$= t + \sum_{m=2}^{\infty} \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} a_m \, t^m = t + \sum_{m=2}^{\infty} A_m t^m .$$

Where $A_m = \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} a_m$.

In our current work we discover some adequate conditions below which the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) = \mathbb{R}_{\alpha,\beta}(t) * \mathcal{H}(t)$

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belonging to the actual classes $S_p(\text{Ꝃ}, \tilde{\alpha})$, $\text{UCV}(\text{Ꝃ}, \tilde{\alpha})$, , and $S^*{}_{\varsigma}$. Following Following lemmas are required for the proof of main results.

Lemma 1.11 [7] A function $\mathcal{H} \in \hat{A}$ is in $UCV(\text{Ꝃ}, \tilde{\alpha})$ if it assure

$$\sum_{m=2}^{\infty} m\{m(1+\text{Ꝃ}) - (\text{Ꝃ} + \tilde{\alpha})\}|a_m| \leq 1 - \tilde{\alpha}.$$

(11)

Lemma 1.12 [7] A function $\mathcal{H} \in \hat{A}$ is in $S_p(\text{Ꝃ}, \tilde{\alpha})$ if it assure

$$\sum_{n=2}^{\infty} m\{m(1+\text{Ꝃ}) - (\text{Ꝃ} + \tilde{\alpha})\}|a_m| \leq 1 - \tilde{\alpha}.$$

(12)

Lemma 1.13 [9] If $\mathcal{H} \in F^{\text{Ꝃ}(\acute{\eta})}{}_{\acute{\varnothing}}$ is explained in (7) , then

$$|a_m| \leq \frac{2|T|(1-\acute{\eta})}{1+\acute{\varnothing}(m-1)} \qquad (13)$$

Lemma 1.14 [11] If $\mathcal{H} \in \hat{A}$ and fulfil

$$\sum_{m=2}^{\infty} (\text{ʒ} + m - 1) \leq \text{ʒ} \qquad , \qquad \text{ʒ} > 0$$

(14)

Then $\mathcal{H} \in S^*{}_{\text{ʒ}}$

Lemma 1.15 [11] If $\mathcal{H} \in \hat{A}$ and satisfy

$$\sum_{m=2}^{\infty} (\text{ʒ} + m - 1) \leq \text{ʒ} \qquad , \qquad \text{ʒ} > 0$$

(15)

Then $\mathcal{H} \in C_{\text{ʒ}}$.

Lemma 1.16 [12] If Ꝃ ∈ N and α̂ ≥ 0, then

(1 + α̂) Ꝃ −1 (Ꝃ − 1)! Γ(1 + α̂) ≤ Γ ((1 + α̂) Ꝃ).

(16)

Lemma 1.17 [13] Suppose $\mathcal{H} \in$ S and have form (1). If for some $\text{Ꝃ}, 0 \leq \text{Ꝃ} < \infty$, the inequality

$$\sum_{m=2}^{\infty} m(m-1)|a_m| \leq \frac{1}{(\text{Ꝃ}+2)} \qquad (17)$$

holds, then $\mathcal{H} \in \text{Ꝃ} - \text{UCV}$. The number $1/\text{Ꝃ} + 2$ cannot be increased.

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Remark 1.18. The circumstances defined in (11), (12), (14) and (15)_ are also essential if $\mathcal{H} \in \hat{A}$ of the form of

$$\mathcal{H}(t) = t - \sum_{m=2}^{\infty} a_m t^m, \qquad \tilde{\alpha}_m \geq 0$$

Main Results

These central results are the connection between the more than a few subclasses of analytic functions by using Rabotnov functions. For additional information about that type of link with Bessel functions and hypergeometric functions see [8, 9, 11, 14,].

Theorem 2.1 Suppose $\mu > 0, \lambda > -1$ and $\tilde{\alpha} \in [0,1)$ with inequality such that

$$\frac{2(1 - \acute{\eta}) \cos Ł}{\acute{\varnothing}} \frac{(1+\tilde{\alpha})}{\tilde{\alpha}} \{K(m-1) + (m - \tilde{\alpha})\} \leq 1 - \tilde{\alpha}.$$

If $\mathcal{H} \in F^{Ł(\acute{\eta})}_{\acute{\varnothing}}$, $\acute{\varnothing} \in [0,1)$ and

$\acute{\eta} < 1$, then the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) \in UCV(K, \tilde{\alpha})$.

Proof. Consider

$$\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) = \mathbb{R}_{\alpha,\beta}(t) * \mathcal{H}(t).$$

$$= t + \sum_{m=2}^{\infty} \frac{\beta^{m-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)m)} a_m z^m \cdot = t + \sum_{m=2}^{\infty} A_m t^m .$$

Where $A_m = \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} a_m$.

To display that the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) \in UCV(K, \tilde{\alpha})$. From Lemma 1.11 we will verify that

$$\sum_{m=2}^{\infty} m\{m(1+K) - (K + \tilde{\alpha})\}|a_m| \leq 1 - \tilde{\alpha}$$

Now

$$\sum_{m=2}^{\infty} m\{m(1+K) - (K+\tilde{\alpha})\} \frac{\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} |a_n|$$

$$\leq 2(1 - \acute{\eta}) \cos Ł \sum_{m=2}^{\infty} m\{m(1+K) - (K+\tilde{\alpha})\} \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} \cdot \frac{1}{1+\acute{\varnothing}(m-1)} \tag{18}$$

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Since, $\frac{1}{1+\acute{\emptyset}(m-1)} \leq \frac{1}{\acute{\emptyset}}$, $\forall_m \geq 2$ therefore (2) becomes

$$\sum_{m=2}^{\infty} m\{m(1+Ꝁ)-(Ꝁ+\tilde{\alpha})\}\frac{\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m|$$

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\{(1+Ꝁ)\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}-(Ꝁ+\tilde{\alpha})\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}\}$$

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\{Ꝁ(m-1)\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}\}+$$

$$(m-\tilde{\alpha})\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}\} \qquad (19)$$

By using the inequalities

$$\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} \leq \frac{1}{(1+\tilde{\alpha})_{m-1}} = \left(\frac{1}{1+\tilde{\alpha}}\right)^{m-1} \quad \text{for all} \quad m \geq 2$$

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\left\{Ꝁ(m-1)\frac{1}{1-\frac{1}{1+\tilde{\alpha}}}+(m-\tilde{\alpha})\frac{1}{1-\frac{1}{1+\tilde{\alpha}}}\right\}$$

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\{Ꝁ(m-1)\frac{1}{1+\tilde{\alpha}}+(m-\tilde{\alpha})\frac{1}{1+\tilde{\alpha}}\}.$$

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\frac{1}{1+\tilde{\alpha}}\{Ꝁ(m-1)+(m-\tilde{\alpha})\}.$$

Equation (19) becomes

$$=\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\frac{1}{1+\tilde{\alpha}}\{Ꝁ(m-1)+(m-\tilde{\alpha})\} \leq 1-\tilde{\alpha}.$$

Hence, the theorem's proof is complete.

Theorem 2.2 Suppose $\lambda > -1, \mu > 0$ and $\tilde{\alpha} \in [0,1)$ with inequality such that

$$\frac{2(1-\acute{\eta})\cos Ꝉ}{\acute{\emptyset}}\frac{1}{5^{n-1},}\frac{(1+\tilde{\alpha})}{\tilde{\alpha}}\{Ꝁ(m-1)+(m-\tilde{\alpha})\} \leq 1-\tilde{\alpha}.$$

If $\mathcal{H} \in Ꝑ^{\tau(\acute{\eta})}_{\acute{\emptyset}}$, $\acute{\eta} < 1$ and $\acute{\emptyset} \in [0,1)$, then the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) \in S_p(Ꝁ,\tilde{\alpha})$.

Proof. Consider

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$$\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t}) = \mathbb{R}_{\alpha,\beta}(t) * \mathcal{H}(t).$$

$$= t + \sum_{m=2}^{\infty} \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} a_m \; t^{m\cdot} = t + \sum_{m=2}^{\infty} A_m t^m .$$

Where $A_m = \frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} \mathrm{a}_m.$

To show that $\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t}) \in$ UCV $(Ҡ, \tilde{\alpha})$. is a convolution operator. We will show from Lemma 1.11 that

$$\sum_{m=2}^{\infty} m\{m(1+Ҡ) - (Ҡ+\tilde{\alpha})\}|a_m| \le 1 - \tilde{\alpha}$$

Now

$$\sum_{m=2}^{\infty} m\{m(1+Ҡ) - (Ҡ+\tilde{\alpha})\}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m|$$

$$\le 2(1 - ή)\cos Ƀ \sum_{m=2}^{\infty} m\{m(1+Ҡ) - (Ҡ+\tilde{\alpha})\}\frac{\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} \cdot \frac{1}{1+ǿ(m-1)}. \quad (20)$$

Since, $\frac{1}{1+ǿ(m-1)} \le \frac{1}{ǿ}$, $\forall_m \ge 2$ therefore (2) becomes

$$\sum_{n=2}^{\infty} m\{m(1+Ҡ) - (Ҡ+\tilde{\alpha})\}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m|$$

$$= \frac{2(1-ή)\cos Ƀ}{ǿ}\{(1+Ҡ)\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)} - (Ҡ+\tilde{\alpha})\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}\}$$

$$= \frac{2(1-ή)\cos Ƀ}{ǿ}\{Ҡ(m-1)\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)}\} + (m-\tilde{\alpha})\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)}\} \quad (21)$$

By using the inequalities

$$\frac{\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})Ҡ)} \le \frac{1}{(1+\tilde{\alpha})_{m-1}(m-1)!} \qquad \forall_m \ge 2$$

Equation (21) becomes

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$$= \frac{2(1-ή)\cos Ⱡ}{ǿ}\{Ꝁ(m-1)\frac{\beta^{m-1}}{(1+\tilde{\alpha})_{m-1}(m-1)!} + (m-\tilde{\alpha})\frac{\beta^{m-1}}{(1+\tilde{\alpha})_{m-1}(m-1)!}.$$

(22)

$$(m-1)! \geq 5^{m-1}, \qquad \forall_m \geq 2$$

and

$$\frac{\beta^{m-1}}{(1+\tilde{\alpha})_{m-1}} \leq \left(\frac{1}{1+\tilde{\alpha}}\right)^{m-1}$$

So equation (22) becomes

$$= \frac{2(1-ή)\cos Ⱡ}{ǿ}\{Ꝁ(m-1)\left(\frac{1}{1+\tilde{\alpha}}\right)^{\tilde{\alpha}-1}\frac{1}{5^{m-1},} + (m-\tilde{\alpha})\left(\frac{1}{1+\tilde{\alpha}}\right)^{m-1}\frac{1}{5^{m-1},}.$$

$$= \frac{2(1-ή)\cos Ⱡ}{ǿ}\frac{1}{5^{m-1},}\left\{Ꝁ(m-1)\frac{1}{1-\frac{1}{1+\tilde{\alpha}}} + (m-\tilde{\alpha})\frac{1}{1-\frac{1}{1+\tilde{\alpha}}}\right\}$$

$$= \frac{2(1-ή)\cos Ⱡ}{ǿ}\frac{1}{5^{n-1},}\left\{Ꝁ(m-1)\frac{1+\tilde{\alpha}}{\tilde{\alpha}} + (m-\tilde{\alpha})\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\right\}$$

$$= \frac{2(1-ή)\cos Ⱡ}{ǿ}\frac{1}{5^{n-1},}\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\{Ꝁ(m-1) + (m-\tilde{\alpha})\}$$

$$\leq 1-\tilde{\alpha}$$

Hence, the theorem's proof is complete.

Theorem 2.3 Suppose, $\mu > 0, \lambda > -1$ and $\tilde{\alpha} \in [0,1)$ with inequality such that

$$\frac{2(1-ή)\cos Ⱡ}{ǿ}\left\{Ʒ\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\right\} \leq Ʒ$$

If $\mathcal{H} \in Ᵽ^{Ꞇ(ή)}{}_{ǿ}$, $ή < 1$, $ǿ \in [0,1)$ and $Ʒ > 0$, then the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) \in S_Ʒ^{\ *}$.

Proof. To prove that the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(t) \in S_Ʒ^{\ *}$, from Lemma 1.14, we will verify that

$$\sum_{m=2}^{\infty}(Ʒ+m-1)\,|A_m| \leq Ʒ$$

Where $\mathrm{A_m} = \frac{\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}a_m.$ for $m \geq 2$

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To verify that the convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t}) \in S_{ʒ}^{*}$. By the Lemma 1.14 we will prove that

$$\sum_{m=2}^{\infty}(ʒ+\mathrm{m}-1)|A_m|$$

$$=\sum_{m=2}^{\infty}(ʒ+m-1)\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m|.$$

$$\leq\frac{2(1-ή)\cos ƀ}{ǿ}\{\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\big((1+\tilde{\alpha})m\big)}+(ʒ-1)\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\big((1+\tilde{\alpha})m\big)}\}$$

By using the inequalities

$$\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}\leq\frac{1}{(1+\tilde{\alpha})_{m-1}}=\left(\frac{1}{1+\tilde{\alpha}}\right)^{\boldsymbol{m}-1}\qquad\forall\mathrm{m}\geq 2$$

$$\leq\frac{2(1-ή)\cos ƀ}{ǿ}\left\{\frac{1}{1-\frac{1}{1+\tilde{\alpha}}}+(ʒ-1)\frac{1}{1-\frac{1}{1+\tilde{\alpha}}}\right\}.$$

$$\leq\frac{2(1-ή)\cos ƀ}{ǿ}\left\{\frac{1+\tilde{\alpha}}{\tilde{\alpha}}+(ʒ-1)\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\right\}.$$

$$=\frac{2(1-ή)\cos ƀ}{ǿ}\left\{ʒ\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\right\}$$

$$\leq ʒ.$$

Hence, the theorem's proof is complete.

Theorem 2.4 Consider $\mu>0,\lambda>-1$ and $\tilde{\alpha}\in[0,1)$ with inequality such that

$$\frac{1+\tilde{\alpha}}{\tilde{\alpha}}\{m^2+\tilde{\alpha}+Ќm^2-Ќ\}\leq 1-\tilde{\alpha}$$

Then $\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t})$ images $\mathcal{H}(\mathrm{t})\in\mathrm{S}$ into $S_p(Ќ,\tilde{\alpha})$.

Proof. Consider

$$\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t})=\mathbb{R}_{\alpha,\beta}(t)*\mathcal{H}(\mathrm{t}).$$

$$=t+\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}a_m\ t^m=t+\sum_{m=2}^{\infty}A_m t^m.$$

Where $A_m=\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}a_m$.

The convolution operator $\mathbb{R}_{\alpha,\beta}\mathcal{H}(\mathrm{t})$ alters $\mathcal{H}(\mathrm{t})\in\mathrm{S}$ into $S_p(Ќ,\tilde{\alpha})$ to show this. Lemma 1.12 will be used to show that

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$$\sum_{m=2}^{\infty} \mathrm{m}\{\mathrm{m}(1+\mathcal{K})-(\mathcal{K}+\tilde{\alpha})\}|\mathrm{a_m}| \leq 1-\tilde{\alpha}.$$

Now

$$\sum_{m=2}^{\infty} m\{\mathrm{m}(1+\mathcal{K})-(\mathcal{K}+\tilde{\alpha})\}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m|$$

$$= \begin{cases} (1+\mathcal{K})\displaystyle\sum_{m=2}^{\infty}\frac{m\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m| \\ -(\mathcal{K}+\alpha)\displaystyle\sum_{m=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})m)}|a_m| \end{cases}$$

$$\leq \begin{cases} (1+\mathcal{K})\displaystyle\sum_{m=2}^{\infty}\frac{m^2\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})n\right)} \\ -(\mathcal{K}+\tilde{\alpha})\displaystyle\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)} \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{m=2}^{\infty}\frac{n^2\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})n\right)} + k\sum_{m=2}^{\infty}\frac{n^2\beta^{n-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})n\right)} \\ -\mathcal{K}\displaystyle\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)} + \tilde{\alpha}\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)}. \end{cases}$$

$$= \begin{cases} (m^2+\tilde{\alpha})\displaystyle\sum_{m=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)}. \\ +\mathcal{K}(m^2-1)\displaystyle\sum_{n=2}^{\infty}\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma\left((1+\tilde{\alpha})m\right)}. \end{cases}$$

By using the inequalities

$$\frac{\beta^{m-1}\Gamma(1+\tilde{\alpha})}{\Gamma((1+\tilde{\alpha})n)} \leq \frac{1}{(1+\tilde{\alpha})_{m-1}} = \left(\frac{1}{1+\tilde{\alpha}}\right)^{m-1} \qquad \forall \mathrm{m} \geq 2$$

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$$\leq \begin{cases} (m^2+\tilde{\alpha})(\frac{1}{1+\tilde{\alpha}})^{m-1} \\ +K(m^2-1)(\frac{1}{1+\tilde{\alpha}})^{m-1} \end{cases}$$

$$= \begin{cases} (m^2+\tilde{\alpha})\frac{1}{1-\frac{1}{1+\tilde{\alpha}}} \\ +K(m^2-1)\frac{1}{1-\frac{1}{1+\tilde{\alpha}}} \end{cases}$$

$$= \begin{cases} (m^2+\tilde{\alpha})\frac{1+\tilde{\alpha}}{\tilde{\alpha}} \\ +K(m^2-1)\frac{1+\tilde{\alpha}}{\tilde{\alpha}} \end{cases}$$

$= \frac{1+\tilde{\alpha}}{\tilde{\alpha}} \{m^2 + \tilde{\alpha} + km^2 - K\} \leq 1 - \tilde{\alpha}$.

This completes the proof of Theorem.

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