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THE DENSITY IMPACT ON THE MATHEMATICAL MODEL OF THE SELF-PROPELLED PARTICLES IN THE HETEROGENEOUS MEDIA

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Abstract
The collective dynamics of the self-driven particles was investigated mathematically and computationally. A two-dimension square shaped heterogeneous medium with boundaries $L = 40$ was developed to study the behavior of the self-propelled particles in the existence of the static obstacles. It was examined that how changes in density (N_b) impact on the collective motion of the SPP's within the Heterogeneous media. The maximum cohesive motion and ferromagnetic alignment of the particles was determined in the presence of optimal noise(η). It was observed that as the number of the particles was increased, exceptional collective alignments among the particles were formed. At $N_b = 1000$ and $N_b = 1400$ an uninterrupted continuous escalation in motion was identified and the value of collective motion of self-propelled particles was reached at $v = 0.43$ out of $v = 1.0$ at $t = 200$ time steps. The inherited order parameter among the particles in this model is an important critical factor.

INTRODUCTION

The self-propelled particles (SPPs) are the self-navigating particles having strength to move independently from place to place without leadership[1] and any external force. The SPPs get their energy from the natural environment. They have group alignment like swarming, flocking and etc. The particles have force of attraction as well as repulsion among themselves. The SPPs have an average speed so that they could be closer to each other during their movement. It protects them from the different predators. The SPPs often make sudden changes in their direction according to environmental changes, predators or an external influence[2]. There are so many applications such as biological system, drugs delivery[3], designing artificial swarms, environmental monitoring like development of the sensors that navigate and collect data from different environments, Robotics [4], Microfluidics, Crowd dynamics, Synthetic biology, Agriculture, Nanotechnology and etc.

The paper is structured in the following way. In first section introduction is given. After that literature review is given which is highlighting relevant theories and previous research. The Methodology section explains the research methods and procedures employed in the study, detailing how data was collected and analyzed. In the Results and Discussion section, the findings are presented and interpreted, with insights drawn from the data. The Conclusion summarizes the overall research, emphasizing the key takeaways and implications. Finally, the References section lists all the sources and literature cited throughout the paper.

1.1 Literature Review:

The first time the self-propelled model was introduced by Reynolds [5] In which moving birds were presented as graphical visualizations. Three types of interaction among the SPPs were presented that included, follow the direction of the neighbors, stay near to the center of the flock, and avoidance of the collisions. The facts about the particles system were simulated and a separated behavior model was formed so that it could work as an animal congregation system with their own individual direction. It was noticed that every particle worked as an independent actor that acted according to its local environment. The average results of the SPPs were simulated in the crowded medium to know about the fundamental habits of the simulated particle [5]. Furthermore, a detail insight on the basic laws and observation of the collective motion of the self-propelled particles was introduced by the Vicsek. In which a macroscopic state in the system of the particles was reported about the transition between the algebraic divergence and the state included discontinuity [4]. A novel type model was also introduced to study the behavior of SPPs in which variation in the different parameters was investigated in the presence of moving obstacles [6]. It was showed that how particles behave when they did interact with optimal noise. The impact of the avoidance radius, obstacle density and noise of the particles was investigated. It was stated that at lower noise levels, significant collective motion and particle alignment in clusters were observed, whereas at higher noise levels, less group alignment was noted and only minimal collective motion was recorded. The system behaved non-monotonic due to changes in the avoidance radius and obstacle density [7]. A research based on the standard Vicsek model investigated that how particles disperse and collective alignment under noise-free and noisy environments. In the noise-free case, a grid-based method exhibited that smaller interaction radii or larger particle quantities lead to weaker velocity correlations and increased particle aggregation at consensus. Under noisy conditions, the result of noise on consensus varies non-monotonically due to unbalanced noise distribution. This research enhances our comprehension of collective motion and suggests future integration of particle-based models with hydrodynamic approaches, as well as extensions to classical aggregation theories [8]. A comparative study of the effects of the interaction radius, noise and speed of the SPP's in the presence of the obstacles and without obstacles was investigated [9]. A great fluctuation in the existence of obstacles and a less fluctuation in the absence obstacles was observed. It was also detected that the optimal noise, which maximized the collective motion of the particles exists in the presence of obstacles [10]. For the life science of complex system of SPPs was also studied and a few new possible future ideas of the SPPs dynamics were extracted and gained the attention of the readers. For a complex living system, a mathematical theory was developed [11]. A numerical and analytical approach was used to investigate the collective dynamics of the SPPs [7]. In addition to this a two-dimensional continuum model was developed to investigate the collective behavior of the self-propelled particles [12], [13], [14], [6] by varying the noise. The declined in the collective motion of SPP is observed when the noise was increased [11]. For the investigation of the transport properties of the moving particles a heterogeneous medium was established [7], [12], [15] and developed a the theory of flock of the birds and to calculate the behavior of the SPPs mathematically and computationally. The natural environmental facts of the SPPs in the existence of the obstacles such predator, wind, temperature and etc. was investigated. A square shaped heterogeneous media was developed in which moving particles and static obstacles were denoted by arrows and dots respectively. The numerous useful findings were observed in the different simulations by restricting the distinct parameters. It was examined that a point exists under which the system had the maximum order parameter among the particles in the presence of ideal noise and there was another point where the function demonstrated a long-range collective motion at the small obstacles density. Two critical point were suggested where the system had a quasi-long ranged order parameter and the system become disorder at the both points in the strong heterogeneous medium [15]. It was also declared that at the smaller noise value, the motion of the particles was diffusive and characterized by the diffusion coefficient [12]. The SPPs (birds) have a habit of tracking the neighbors' motion whose movements are more salient. This ability is applied to the robotic swarms to enhance the collective dynamics of robots, which is improved during escape smoothly together. This emerging approach to studying the collective dynamics of self-propelled particles is establishing a significant bridge between natural animal behavior and robotic swarm systems. It was demonstrated that the motion salience has an important influence on how birds coordinate movement. When it comes to swarm robotics, adaptive MS-based interaction enhances group behavior, especially in confined environments. This expands our insight of self-organization and provides constructive advice for developing more effective robotic swarms [16].

2. Methodology

First time Vicsek et al [17] introduced the fundamental approach to the modelling of the self-propelled particles. He examined a non-homogeneous equilibrium system of clustering, transporting and phase transition of the SPP's. The particles speed was recognized by the fundamental principles and arbitrary function. A static obstacle approach is used with following the central direction of the particles with the radius "r" from their surroundings. A non-equilibrium analogue of transportation such as the ferromagnetic type model of transport among the SPP's is demonstrated. The collective motion of the SPP's is an important aspect in this model in which rule alignment, identical direction and the perturbation level is applied by the temperature. It is an exciting model with the organic application such as migration, microorganism, crowed dynamics, drugs delivery, Synthetic biology and Nanotechnology.

A squared figured linear plane model with the border surroundings is created to get the simulations in which moving particles and static obstacles are denoted by arrows and dots respectively. The radius is changed by the time Δt. Where $\Delta t = 1$ $r = 1$ are time intervals between two updated positions of the SPP's. The ith particle position is updated by the equation.

$$x_i(t+1) = x_i(t) + v_i(t)\Delta t \qquad (1)$$

Equation (1) has been taken from [9], here x_i denote the ith particle position, v_i is the absolute speed of the ith object and change is the time by Δt. The particles speed is changed by the time from $t = 0$ and onward. The angle of the particles is measured as $\theta(t+1) = \langle \theta(t) \rangle_r + \Delta\theta$ Where the $\theta(t+1)$ is an angle of the particles. The average direction of the SPPs is denoted by $\langle \theta(t) \rangle_r$, here "r" is circular Radius of the SPPs and $\Delta\theta$ variable is used as the noise in the form of temperature. The particles average direction is measured by $\langle \theta(t) \rangle_r = \left[\frac{\langle \sin(\theta(t)) \rangle_r}{\langle \cos(\theta(t)) \rangle_r} \right]$

This Equation has been taken from[9]. The velocity of particles is not static, so it can be determined by the interaction among the particles. Two basic parameters η and ρ are used to find the density as $p = \frac{N}{L^2}$, Here the p is the density of the particles, η is the noise, L is the dimension of the heterogeneous medium and N is the particles density. A detail study about the SPP's dynamics is has been done. The velocity of complete system of the particles is computed by $v_a = \frac{1}{N_v}\left|\Sigma_{i=1}^{N} vi\right|$, if $v \to 0$ particles do not have any motion and if $v \to \infty$ the ferromagnetic behavior failed. Thus, the velocity of the SPP's is zero when direction of the moving particles is depressed at random. At the static noise, its stated that how the velocity is increases and the group alignment of the SPPs are increased. Here it is exhibited the collective motion of the SPPs

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in the presence of the a weak, a medium and a strong density. To manage the speed of the particle's temporal differential equation $\dot{x}_i = v_o v(\theta_i)$ is used. Here v_o is active particles velocity, x_i is the resultant location of the ith objects and $v(\theta_i) = (\cos(\theta), \sin(\theta))^t$. With the help of the temporal derivative, velocities of the objects are determined. For changing the direction of the particles, a differential equation

$$\dot{\theta}_i = g(x_i)\left[\frac{\gamma_b}{n_b(x_i)} \sum_{|x_i - x_j| < R_b} \sin(\theta_i - \theta_j)\right] + h(x_i) + n\varepsilon_i(t) \qquad (2)$$

Equation (2) has taken from [9], [14], here equation (2) is used to manage the direction of the particles, a function $g(x_i)$ is used to prevent the collision of the particles with the obstacles and keep at the distance from themselves. For the great alignment, the collaboration limit R_b and the angular relaxation velocity γ_b the SPPs are measured.

$n_b(x_i)$ and $n_o(x_i)$ are associated to the obstacles in the heterogeneous media which presence at the space less equivalent than $R_b[R_o]$ from x_i.

The interaction between particles and obstacles are defined by the function $h(x_i)$,

$$h(x_i) = \begin{cases} \dfrac{\gamma_o}{n_o(x_i)} \displaystyle\sum_{|x_i - y_k| < R_0} \sin(\alpha_{k,i} - \theta_i) & \text{if } n_0(x_i) > 0 \\ 0 & \text{if } n_o(x_i) = 0 \end{cases} \qquad (3)$$

Equation (3) has been taken from[9], [14], Where y_o and R_o are the collaboration among particles and obstacles. The function $a_{k,i}$ is an angle of polar coordinates of $x_i - y_k$ *and* θ_i is the direction of the particles motion. The obstacle intensification through a small movement is characterized as

$$r = \langle r(t) \rangle_t = \left\langle \left| \frac{1}{N_b} \sum_{i=1}^{N_b} e^{i\theta_i(t)} \right| \right\rangle_t \qquad (4)$$

Equation (4) has been taken from[9], [14] For getting the temporal average **r** is used and that is very convenient to detect the intermediate change in the time.

In this research, the order parameters and alignment inherit of the SPPs are investigated mathematically and computationally. Wherein the diverse behaviors at the distinct parameters of the SPP are found by applying the temporal derivative and other equations. In order to get precise outcome, mathematical and computational approach was used in the Fortran programming to generate the data of the collective dynamics of the SPP. For the visualization GNUPLOT software was used to visualize and interpret the results.

Table.1:Parameters used in the simulation

Symbols	Description
L	Box Length
N_b	Number of the particles
t	Time
η	Noise amplitude
n_o	Number of obstacles
r	Interaction radius among particles
R_0	Interaction radius between the particles and the obstacles
v	Speed of the particle
Δt	Time interval
γ_o	Particle's turning speed

3. Results and Discussion:

The length of the heterogeneous media was fixed at $L = 40$, number of particles was $N_b = 2000$, time step $t = 200$, number of static obstacles $n_o = 30$, noise amplitude $\eta = 0.01$, Interaction radius between the particle and the obstacles $R_o = 0.5$, Particle's turning speed when it interacts with obstacle $\gamma_o = 1$ The interaction radius between the particles $r = 1.0$, time interval $\Delta t = 0.1$ and absolute velocity $v = 1.0$.

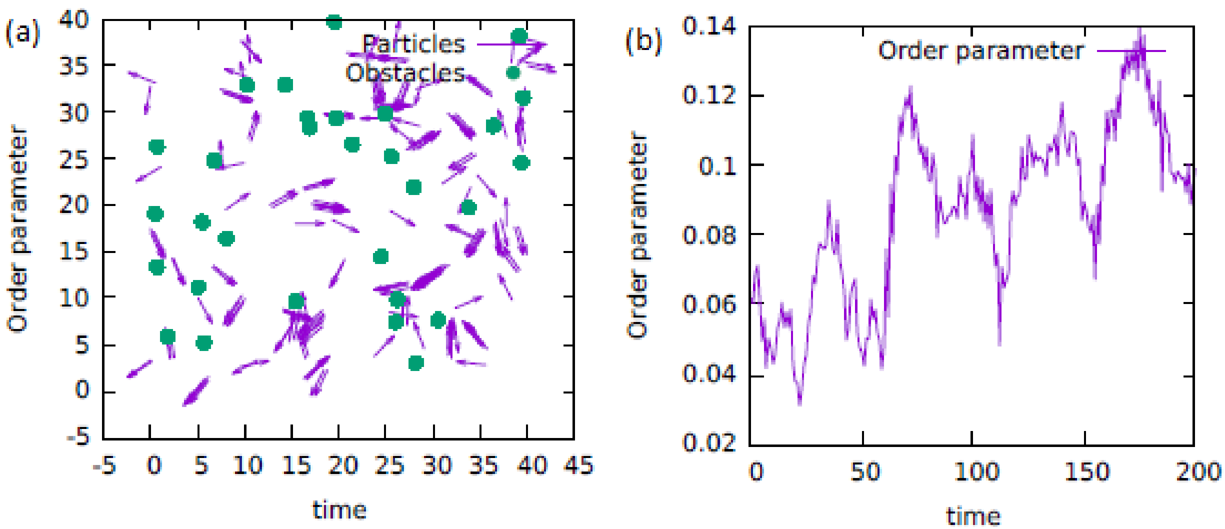

Fig. 1. Particles representing motion instability. $L = 40, t = 200, n_b = 30, \eta = 0.01, R = 0.5,$ $r = 1.0, v = 1.0,$ and $N_b = 200$

In the figures(1-a) green dots demonstrate the static obstacles and the purple arrows represent the moving particles in the heterogeneous medium. In the fig(1-b) purple line graph represents the order parameter of the self-propelled particles in all simulations of the collective motion of the Self-propelled particles was examined by altering particles compactness from very small to a large density level. The order parameter and collective dynamics of the SPPs was Scrutinized in the presence of 30 static obstacles, within the 200-time steps. Initially at $N_b = 10$ it was evident that at the low density a low group alignment among the SPPs was observed. There was an extremely elevated order parameter at the very small number of the particles was witnessed. From time step $t = 175$ to $t = 200$ a great number of fluctuations and unstable order parameter was evaluated fig(1-b). As the particles density was slowly increased, a random motion of the particles with weak alignment was observed. There was a less cohesive and individual behavior was realized. The particles avoided from the obstacles. At $N_b = 200$ density, there were fluctuations in the collective dynamics with an average order in the motion was assessed in the Fig.(1).

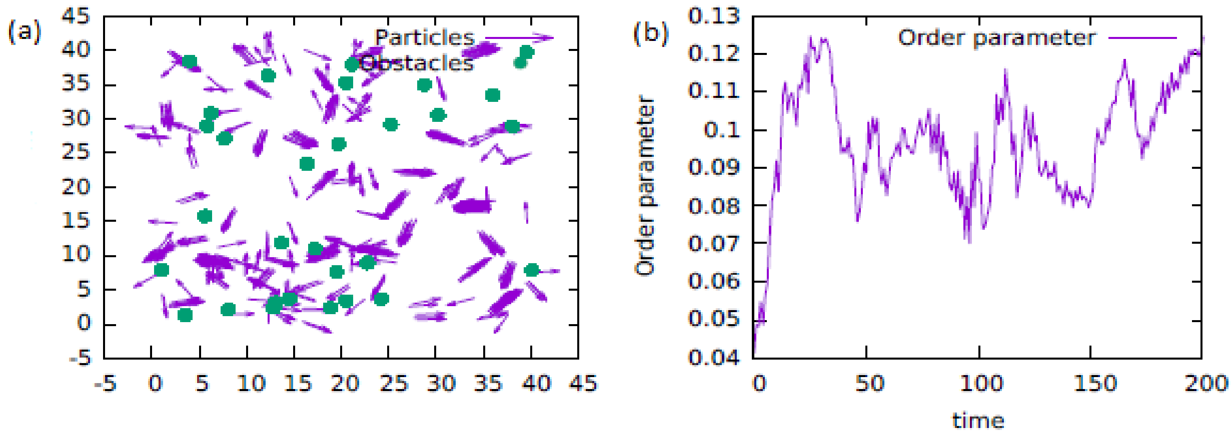

Fig. 2. Particles exhibiting unstable order parameter. $L = 40, t = 200, n_b = 30, \eta = 0.01, R = 0.5, r = 1.0, v = 1.0$, and $N_b = 500$

At the $N_b = 500$ it was noticed that as the number of the particles were increased, the small groups alignments and fluctuated motion was noticed in the fig(2-a & b).

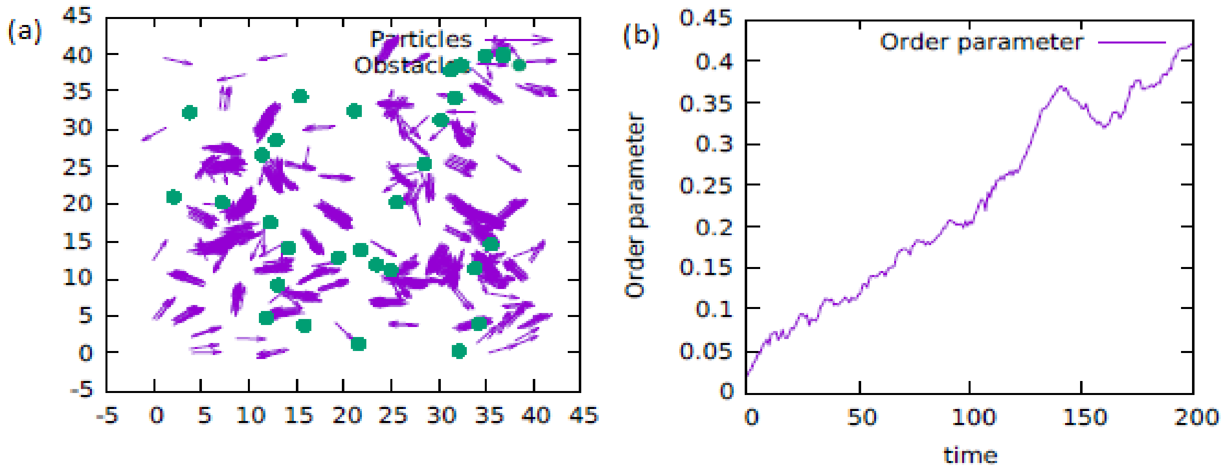

Fig. 3. Particles exhibiting stable collective dynamics. $L = 40, t = 200, n_b = 30, \eta = 0.01, R = 0.5, r = 1.0, v = 1.0$, and$N_b = 1000$

It was witnessed in the fig.(3-a), that a great group alignment was formed as the self-propelled particles reached up to the $N_b = 1000$. Each group has an exceptional direction with the random collective motion. The ferromagnetic force exists among the SPPs. The particles were Struggling to evade from the obstacles and it prevents the particles from their collision with obstacles. In the fig.(3-b) as the time steps were increased the order parameter around $v = 0.42$ out of $v = 1.0$ of the self-propelled particles were increased rapidly with very minor fluctuations were distinguished.

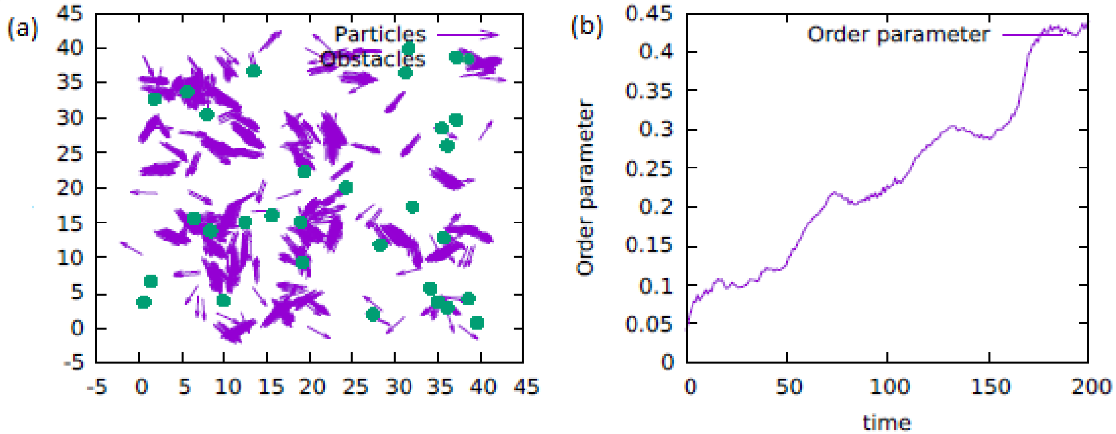

Fig. 4. Particles demonstrating consistent order parameter. $L = 40$, $t = 200$, $n_b = 30$, $\eta = 0.01$, $R = 0.5$, $r = 1.0$, $v = 1.0$, and $N_b = 1400$

In this simulation an extensive groups formation with well-maintained and uninterrupted collective motion was noted. At $\mathrm{N_b} = 1400$, a comparable behavior was evident that with the swift increase in particles count, a dense groups arrangement of the particles was escalated quickly fig(4-a). In each herd there was an average direction of the self-propelled particles. As the time steps were increased an impressive order parameter of the self-propelled particles without fluctuations was recorded. At $\mathrm{t} = 200$ the order parameter of the self-propelled particles reached at $\mathrm{v} = 0.45$ out of the $\mathrm{v} = 1.0$. it was an uninterrupted continuous escalation in motion was identified fig(4-b).

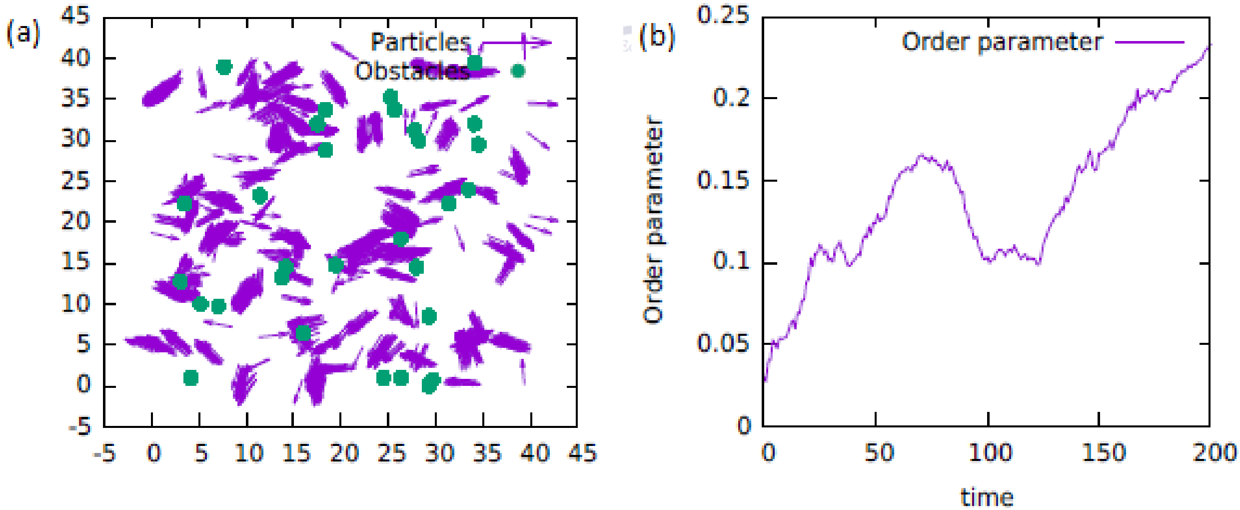

Fig. 5. Particles exhibiting the formation of interactive groups. $L = 40$, $t = 200$, $n_b = 30$, $\eta = 0.01$, $R = 0.5$, $r = 1.0$ $v = 1.0$, and $N_b = 1800$

At $\mathrm{N_b} = 1800$, in the simulation that there was an strong cluster forming of the particles were noticed. Each cluster, it was recorded a moderate direction towards their place of arrival. At the initial time steps a small order parameter was noticed but as the time steps were increased a slow raise in motion was detected. Between $\mathrm{t} = 100$ to $\mathrm{t} = 135$ a moderate but stable motion was recorded. After $\mathrm{t} = 135$, a continuous raised in the order parameter was noted in fig (5-b).

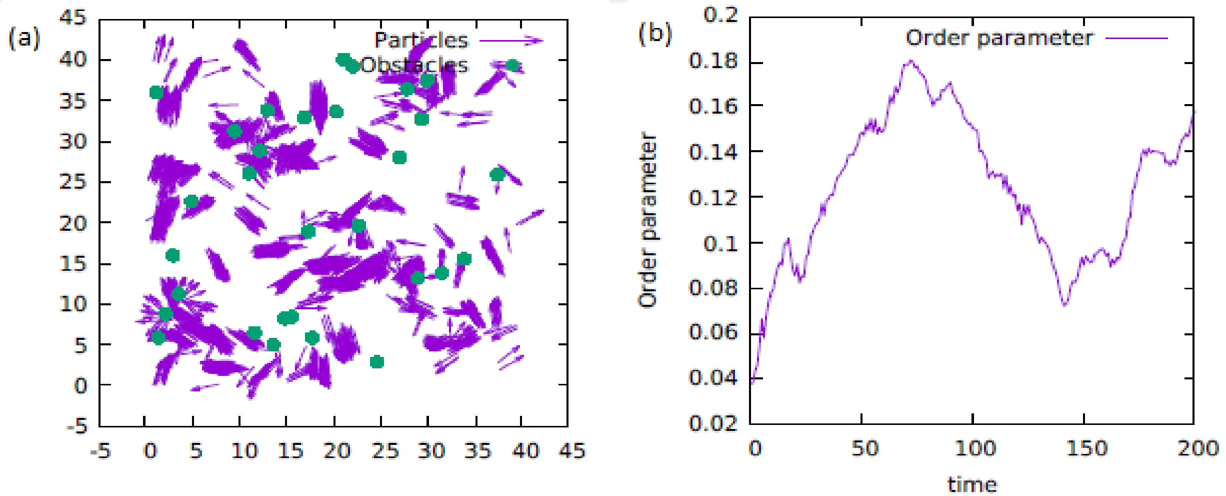

Fig. 6. Particles showing group alignment. $L = 40$, $t = 200$, $n_b = 30$, $\eta = 0.01$, $R = 0.5$, $r = 1.0$, $v = 1.0$, and $N_b = 2000$

At $N_b = 2000$, the system showed a great phenomenal group alignment with a Small-scale movement can be seemed fig (6-1). Flocks of the particles have an average direction towards their destination. The particles within the flock exhibited a ferromagnetic force that has protected them from the predictor like static obstacles exist in the heterogeneous media and kept them near to the flock's centre. At the initial time steps a slow but rapid increase in the order parameter was noticed till $t = 70$ after that a sudden quick decline in the particles motion till $t = 145$ was observed. Precisely a low scale ups and down in the collective motion of the self-propelled particles was showed in the fig(6-b).

4. Conclusion

Collective motion of SPPs was investigated in the existence of static obstacles. A two-dimension continuum heterogeneous medium was developed to simulate the behavior of the SPPs. For this purpose a mathematical and computational approach with simple equations-based model was used. The density impact on the SPPs behavior was examined. In this study, weak, medium, and strong density conditions were simulated. It was found that at the lower density, the less cohesive movement of the self-propelled particles without group alignment was witnessed. With the increasing particle density such as $N_b = 1000$ and $N_b = 1400$, SPPs started showing alignment in their direction. The value of the order parameter appeared to be equal to 0.43 and 0.45 respectively. It was examined that individual groups had an identical direction towards their destination. These findings will significantly contribute to our comprehension of natural laws and strategies in the self-propelled particles during the navigation and migration journey.

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